EE 232 Lightwave Devices
Lecture 6: Optical Waveguide

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Reading: Chuang, Chap 7

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Maxwell Wave Equation

In free space

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$D = \varepsilon E$$

$$B = \mu H$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} + \mu_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

In most cases, E field has sinusoidal temporal component:

$$E \sim e^{-i\omega t}$$

$$\nabla^2 \mathbf{E} + \mu_0 \varepsilon \omega^2 \mathbf{E} = 0$$

$$\left(\nabla^2 + \mu_0 \varepsilon \omega^2 \right) \mathbf{E} = 0$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8 \text{ m/s}$$

Transverse Electro-Magnetic (TEM) Wave

Solution in uniform media:

$$\mathbf{E} = \hat{y} E_0 e^{i\beta z - i\omega t}$$

$$\beta = \sqrt{\mu_0 \varepsilon} \cdot \omega = \frac{\omega}{c/n} = n\frac{\omega}{c} = n\frac{2\pi}{\lambda} = nk_0$$

Once E field is solved, H field can be found by

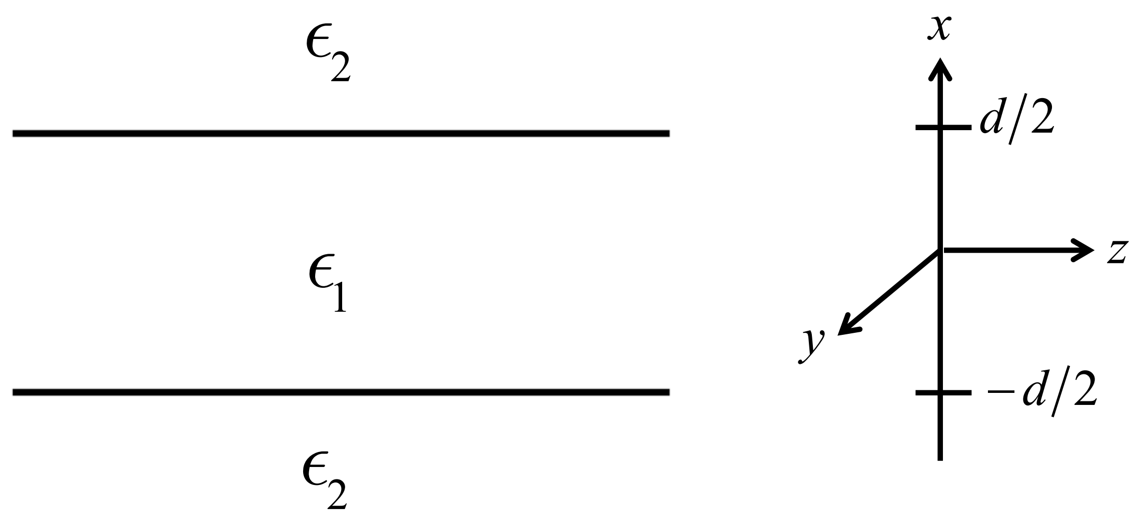
$$\nabla \times E = -\frac{\partial B}{\partial t} = i\omega B = i\omega\mu_0 H$$

$$H = \frac{1}{i\omega\mu_0}\nabla \times E$$

E_y is the only non-zero component:

$$\mathbf{H} = \frac{1}{i\omega\mu_0}\left(-\hat{x}\frac{\partial E_y}{\partial z} + \hat{z}\frac{\partial E_y}{\partial x}\right) = -\hat{x}\frac{i\beta}{i\omega\mu_0}E_0 e^{i\beta z - i\omega t} = -\hat{x}\frac{\beta}{\omega\mu_0}E_0 e^{i\beta z - i\omega t}$$

Slab waveguide

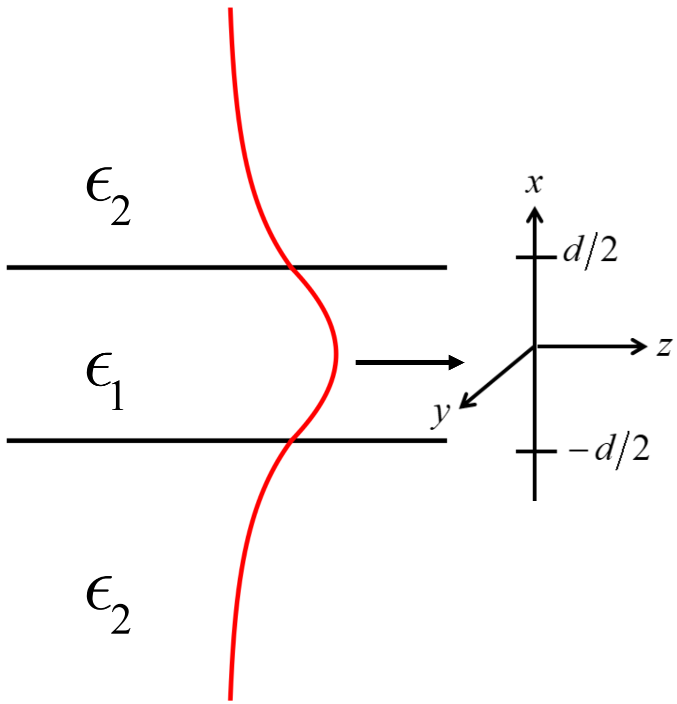


Slab waveguide consists of a slab of high-index material surrounded by low-index material ($\epsilon_1 > \epsilon_2$). The waveguide is assumed to be infinitely large in the y and z-directions.

We wish to find confined electromagnetic modes that propagate in the +z direction and solve the source-free time-harmonic wave equation

$$\left(\nabla^2 + \omega^2 \mu \epsilon\right)\mathbf{E} = 0$$

Slab waveguide



TEM wave does not exist in slab waveguide. Slab waveguides support Transverse Electric (TE) and Transverse Magnetic (TM) modes. First, let's look at TE mode:

$$\mathbf{E} = \hat{y}E_y(x,z) \rightarrow E_y(x,z) = f(x)h(z)$$

where we assume there is no dependence on y given the slab is translationally invariant in the y-direction. Along the z-direction we expect a traveling wave solution

$$h(z) = C_1 e^{-i\beta_z z} + C_2 e^{i\beta_z z}$$

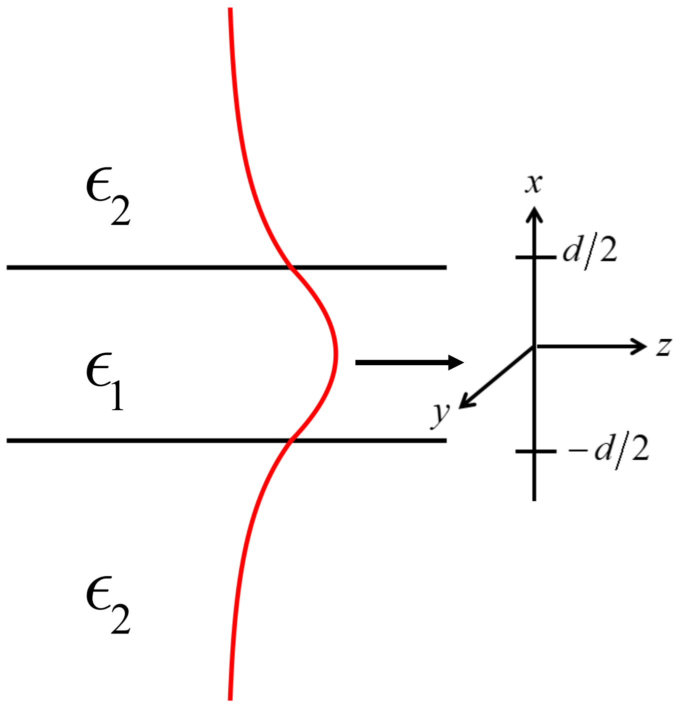
Maxwell's wave equation becomes

$$\left(\nabla^2 + \omega^2 \mu \epsilon\right) E_y(x,z) = 0$$

Note this is exactly the same as the Schrodinger Eq. for QW:

$$\left[\frac{-\hbar^2}{2m_e^*}\nabla^2 + V(\mathbf{r})\right]\psi_{env}(\mathbf{r}) = E\psi_{env}(\mathbf{r})$$

Slab waveguide



Similar to QW solution, along the x-direction, we expect a standing wave solution in the waveguide core and evanescent solution in the cladding.

Even solution

$$f(x) = \begin{cases} A_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ A_2 \cos(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Odd solution

$$f(x) = \begin{cases} B_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ B_2 \sin(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Slab waveguide

① Plug into wave equation →

$$\beta_x^{\ 2} + \beta_z^2 = \omega^2 \mu_1 \epsilon_1$$
$$-\alpha^2 + \beta_z^2 = \omega^2 \mu_2 \epsilon_2$$

② Apply boundary conditions at interface between core and cladding. Tangential component of electric and magnetic field are equal across interface.

$$E_{y,core}\Big|_{x=\pm\frac{d}{2}} = E_{y,clad}\Big|_{x=\pm\frac{d}{2}}$$
$$H_{z,core}\Big|_{x=\pm\frac{d}{2}} = H_{z,clad}\Big|_{x=\pm\frac{d}{2}}$$

→

(even) $\alpha = \frac{\mu_2}{\mu_1}\beta_x \tan\left(\beta_x \frac{d}{2}\right)$

(odd) $\alpha = -\frac{\mu_2}{\mu_1}\beta_x \cot\left(B_x \frac{d}{2}\right)$

Slab waveguide

③ After rearranging

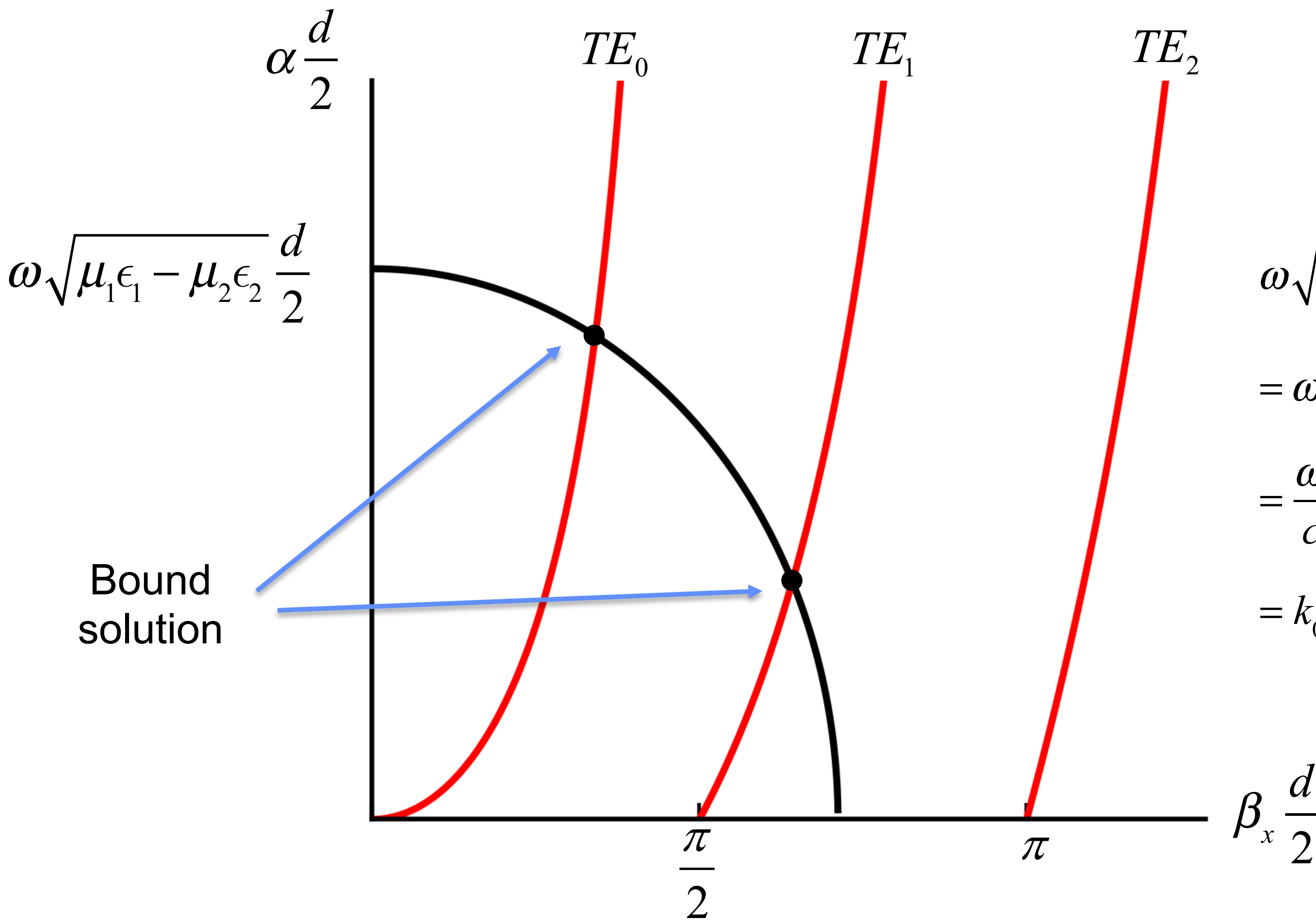
$$\left(\beta_x \frac{d}{2}\right)^2 + \left(\alpha \frac{d}{2}\right)^2 = \omega^2 \left(\mu_1 \epsilon_1 - \mu_2 \epsilon_2\right)\left(\frac{d}{2}\right)^2$$

$$\left(\alpha \frac{d}{2}\right) = \frac{\mu_2}{\mu_1}\left(\beta_x \frac{d}{2}\right)\tan\left(\beta_x \frac{d}{2}\right) \qquad \text{(Even)}$$

$$\left(\alpha \frac{d}{2}\right) = -\frac{\mu_2}{\mu_1}\left(\beta_x \frac{d}{2}\right)\cot\left(B_x \frac{d}{2}\right) \qquad \text{(Odd)}$$

Slab waveguide

④ Solve graphically



$$\omega\sqrt{\mu_1\epsilon_1 - \mu_2\epsilon_2}\,\frac{d}{2}$$
$$= \omega\sqrt{\mu\epsilon_0}\sqrt{\epsilon_{r1} - \epsilon_{r2}}\,\frac{d}{2}$$
$$= \frac{\omega}{c}\sqrt{n_1^2 - n_1^2}\,\frac{d}{2}$$
$$= k_0\sqrt{n_1^2 - n_1^2}\,\frac{d}{2}$$

Cutoff condition

In the example on the previous slide we see that the TE_1 mode would not have a solution and would be "cutoff" if the radius of the circle is less than $\pi/2$. The cutoff condition for each mode can be generalized as

$$k_0 \frac{d}{2}\sqrt{n_1^2 - n_1^2} = m\frac{\pi}{2} \qquad m = 0,1,2,3...$$

(Cutoff condition for TE_m mode)

The waveguide will be single-mode if all modes except the fundamental mode are cutoff.

$$k_0 \frac{d}{2}\sqrt{n_1^2 - n_1^2} < \frac{\pi}{2}$$

(Single mode condition)

Effective index

Effective index $n_{eff} = \frac{\beta_z}{\beta_0} \qquad \beta_0 = \frac{2\pi}{\lambda_0}$

High-frequency limit

Radius $\rightarrow \infty$ as $\omega \rightarrow \infty$

$\therefore \alpha \rightarrow \infty$

$$\beta_z^2 = \omega^2 \mu_1 \epsilon_1 - \beta_x^2 \simeq \omega^2 \mu_1 \epsilon_1 \qquad n_{eff} = \frac{\beta_z}{\beta_0} = \frac{\sqrt{\mu_1 \epsilon_1}}{\sqrt{\mu_0 \epsilon_0}} = n_1 \quad \text{for } \mu_1 = \mu_0$$

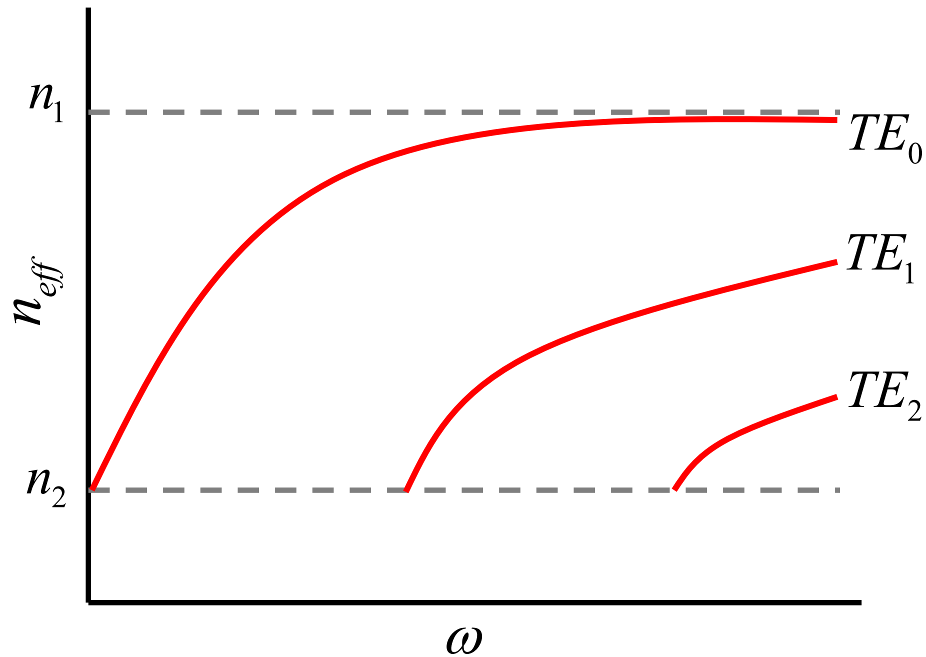
Low-frequency limit

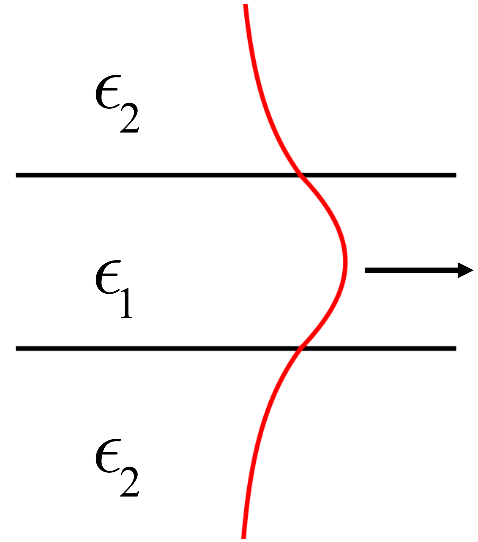
Radius $\rightarrow 0$ as $\omega \rightarrow 0$

$\therefore \alpha \rightarrow 0$

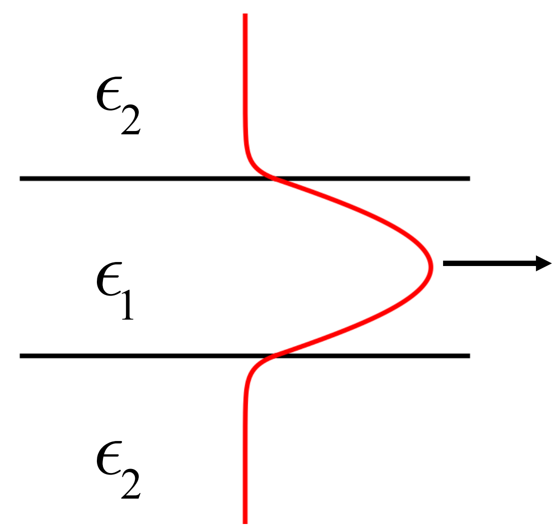
$$\beta_z^2 = \omega^2 \mu_2 \epsilon_2 + \alpha^2 \simeq \omega^2 \mu_2 \epsilon_2 \qquad n_{eff} = \frac{\beta_z}{\beta_0} = \frac{\sqrt{\mu_2 \epsilon_2}}{\sqrt{\mu_0 \epsilon_0}} = n_2 \quad \text{for } \mu_2 = \mu_0$$

Effective index





Low ω



High ω

Effective index is a measure of how confined the mode is to the core

Optical confinement factor

$$\Gamma = \frac{\text{Power in core}}{\text{Total power in mode}} = \frac{\frac{1}{2}\int_{core} \mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{z}\,dx}{\frac{1}{2}\int_{total} \mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{z}\,dx}$$

Weak guidance limit (mode is mostly within cladding)

$$\Gamma \simeq 2\left(\frac{\pi d}{\lambda_0}\right)^2\left(n_1^2 - n_2^2\right)$$

For largest possible Γ :

(1) Thick core

(2) Small wavelength

(3) Large index contrast

TM modes

$$\mathbf{H} = \hat{y}H_y(x,z) \rightarrow H_y(x,z) = f(x)h(z)$$

Along the z-direction we expect a traveling wave solution

$$h(z) = C_1 e^{-i\beta_z z} + C_2 e^{i\beta_z z}$$

Along the x-direction, we expect a standing wave solution in the waveguide core and evanescent solution in the cladding.

$$f(x) = \begin{cases} A_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ A_2 \cos(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Even solution

$$f(x) = \begin{cases} B_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ B_2 \sin(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Odd solution

Eigenequations:

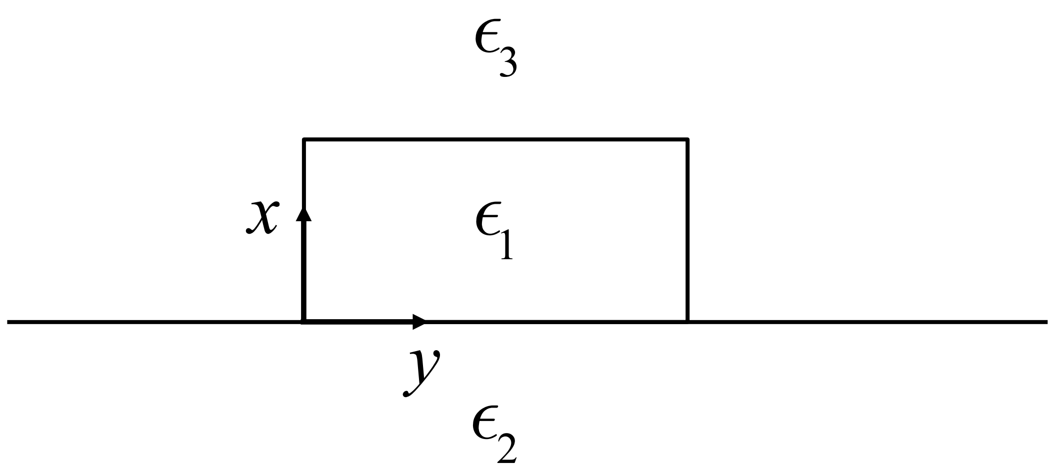
$$\left(\beta_x \frac{d}{2}\right)^2 + \left(\alpha \frac{d}{2}\right)^2 = \omega^2 \left(\mu_2\epsilon_2 - \mu_1\epsilon_1\right)\left(\frac{d}{2}\right)^2$$

$$\left(\alpha \frac{d}{2}\right) = \frac{\epsilon_2}{\epsilon_1}\left(\beta_x \frac{d}{2}\right)\tan\left(\beta_x \frac{d}{2}\right)$$ Even solution

$$\left(\alpha \frac{d}{2}\right) = -\frac{\epsilon_2}{\epsilon_1}\left(\beta_x \frac{d}{2}\right)\cot\left(B_x \frac{d}{2}\right)$$ Odd solution

Rectangular waveguides

Rectangular waveguides have dielectric contrast in two-directions



Rectangular waveguides do not support pure TE or TM modes!
Instead they support hybrid modes.

Rectangular waveguides

Hybrid modes

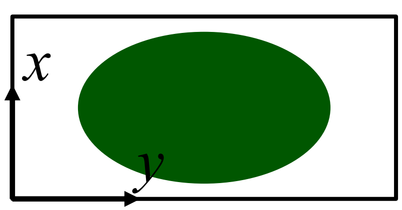
HE_{pq} $\quad H_x, E_y$ are the dominant components (quasi-TE)

EH_{pq} $\quad E_x, H_y$ are the dominant components (quasi-TM)

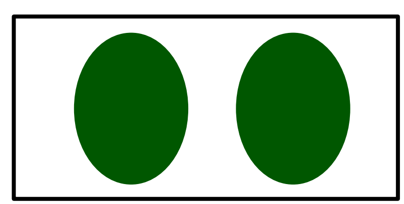
Intensity patterns

$p \rightarrow$ number of nodes in the x-direction

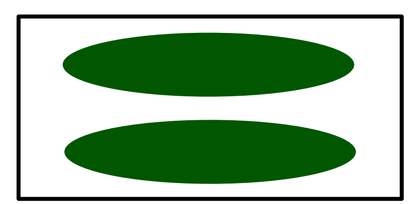
$q \rightarrow$ number of nodes in the y-direction



HE_{00} or EH_{00}

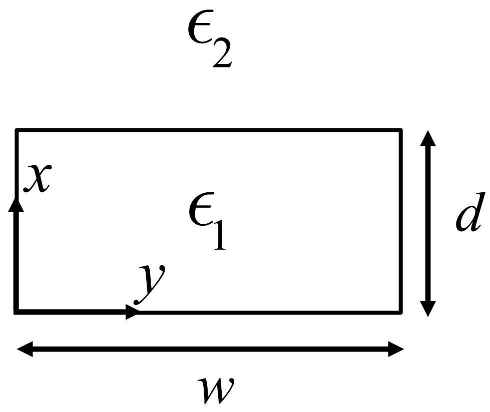


HE_{01} or EH_{01}



HE_{10} or EH_{10}

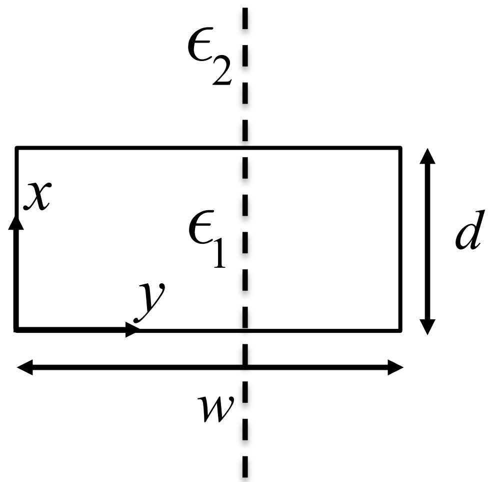
Effective index method



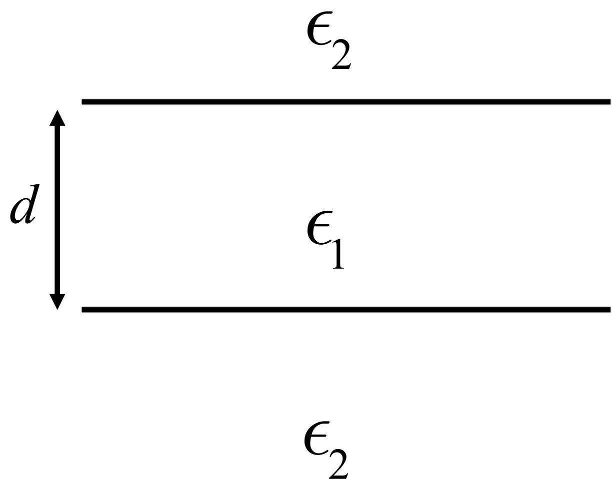
We estimate the propagation constant of the HE_{00} mode with the effective index method. We essentially break the 2D problem into a 1D slab waveguide problem.

To simplify this problem we assume that the waveguide is completely surrounded by the same index. More sophisticated examples are found in the book.

Step 1

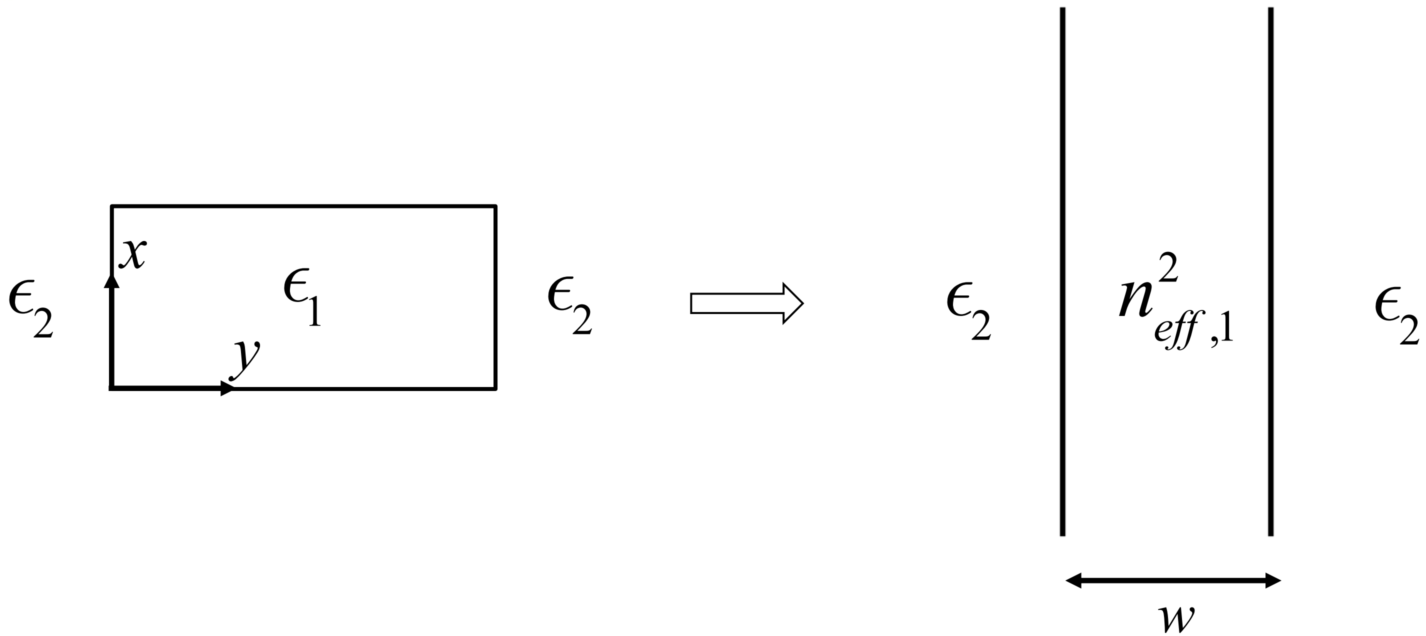


Solve for the TE mode of the slab waveguide with core of permittivity ϵ_1 and cladding with permittivity ϵ_2



Calculate the effective index $n_{eff,1}$ and modal distribution $F(x)$

Step 2



Solve for the TM mode slab waveguide with core of permittivity n_{eff}^2 and cladding with permittivity ϵ_2. Calculate the propagation constant β_z and modal distribution $G(y)$.

The overall propagation constant of the 2D waveguide is then β_z and the modal distribution of the 2D waveguide is given by

$$E_y(x,y) = F(x)G(y)$$